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PERIODIC SOLUTION OF A NONLINEAR PARABOLIC EQUATION INVOLVING BESSEL'S OPERATOR

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Abstract—Standard compactness arguments are used to prove the existence of periodic solution of a nonlinear parabolic equation involving Bessel's operator and associated with a mixed condition. Numerical results are given.

1. INTRODUCTION

We will consider the following initial and boundary problem:

$$u_t - a(t)\left(u_{rr} + \frac{1}{r}\,u_r\right) = f(u), \qquad (t,r) \in (0,T) \times (0,1), \tag{1.1}$$

$$u_r(t,1) + h(t)u(t,1) = 0, \qquad u_r(t,0) = 0, \tag{1.2}$$

$$u(0,r) = u(T,r), \tag{1.3}$$

where the functions $h(t)$ and $a(t)$ are assumed to be real and T-periodic in t. The assumptions on the function $f(u)$ and other properties of the functions $h(t)$ and $a(t)$ needed for our purpose will be specified later.

The physical interpretation of the problem with $f(u) = f(t,r)$ is that of a periodic heat flow in an infinite cylinder with the assumption that the cylinder is subjected to a convective heat transfer (periodic in time) at the boundary surface ($r = 1$) at zero temperature. Inside the cylinder there are circular symmetric sources of heat that change periodically.

The problem with $f(u) = f(t,r)$ was first stated by Minasjan [1] who gave for this problem a classical solution using Fourier Transforms. This method leads to an infinite pseudoregular system of linear algebraic equations. With $f(u) = f(t,r)$ Lauerova [2] proved the existence of a weak solution of this problem by using the Faedo-Galerkin method. Assuming that $f \in C^1(\mathbb{R},\mathbb{R})$ and $f' \leq \varepsilon$ for $\varepsilon > 0$ "small," and using the Faedo-Galerkin method, we prove the existence and the uniqueness of a weak solution of the problem (1.1)–(1.3). If we call $u = u(a,h)$ such a solution, this solution depends continuously on the functions a and h. Note that in this paper, we shall consider the equation (1.1) as an ordinary differential equation in a Banach space for $u(t)$ which stands for $u(t,r)$, so that we shall write $u'(t)$ for $u_t(t,r)$.

2. *A PRIORI* ESTIMATES

Denote by H the real Hilbert space with the scalar product

$$(u,v) = \int_0^1 r u(r)\, v(r)\, dr, \tag{2.1}$$

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and denote by V the real Hilbert space with the scalar product

$$(u,v)_V = \int_0^1 r[u'(r)\,v'(r) + u(r)\,v(r)]\,dr \tag{2.2}$$

with derivatives in the sense of distributions. The norms in H and V induced by the corresponding scalar products are denoted by $\|\cdot\|$ and $\|\cdot\|_V$, respectively. V is continuously and densely embedded in H; identifying H with H^* we have $V \circlearrowleft H \circlearrowleft V^*$.

LEMMA 1. *The imbedding $V \circlearrowleft H$ is compact.*

PROOF. Let the sequence of opens $A_m = (0, 1/m+1)$ and let $W = C^1([0,1])$. W is dense in V. Define $\Gamma^*(m)\,(m \in N)$ by:

$$\Gamma^*(m) = \{\sup_{\|u\|_V=1} \|u\|_{H(A_m)},\ u \in W\}.$$

According to the theorem of Edmunds and Evans [3] , to demonstrate that the embedding $V \circlearrowleft H$ is compact, it is sufficient to prove that $\Gamma^*(0) = 0$.

Let us consider $u \in W$ such that $\mathrm{supp}(u) \subset A_m$. We have

$$u(t) = -\int_t^{1/m} u'(s)\,ds.$$

By the Cauchy-Schwarz inequality, we deduce that

$$|u(t)|^2 \le \left(\frac{1}{m} - t\right) \int_t^{1/m} |u'(s)|^2\,ds.$$

Therefore,

$$\|u\|^2_{H(A_m)} = \int_0^{1/m+1} tu^2(t)\,dt \le \int_0^{1/m} \int_t^{1/m} |u'(s)|^2 \left(\frac{1}{m} - t\right) t\,ds\,dt$$

It folllows from the last inequality , after inverting the variables t and s, that

$$\|u\|^2_{H(A_m)} \le \int_0^{1/m} su'(s)^2\,ds \int_0^s \left(\frac{1}{m} - t\right) \frac{t}{s}\,dt,$$

which implies

$$\|u\|^2_{H(A_m)} \le \int_0^1 s|u'(s)|^2\,ds \int_0^s \left(\frac{1}{m} - t\right) dt \le \|u\|^2_V \left(\frac{1}{2m^2}\right)$$

Thus

$$\Gamma^*(m) \le \frac{1}{2m^2} \to 0 \quad \text{as } m \to \infty,$$

which completes the proof of Lemma 1. ∎

Further, denote $X = L^2(0,T;V)$, $X^* = L^2(0,T;V^*)$. For the dual pairing between X and X^* we use the notation $\langle g, v\rangle$, $g \in X^*$, $v \in X$. The scalar product on the Hilbert space X is denoted by

$$((g,v)) = \int_0^T (g(t), v(t))\,dt, \qquad g, v \in L^2(0,T;H).$$

We shall make the following assumptions:

$$\begin{aligned} &\text{(i)} \quad a(t), h(t),\ T\text{-periodic in } t \quad (a(0) = a(T); h(0) = h(T));\\ &\text{(ii)} \quad a(t) \ge a_0 > 0, \quad h(t) \ge h_0 > 0;\\ &\text{(iii)} \quad h', a' \in L^\infty(\mathbb{R}). \end{aligned} \tag{2.3}$$

$$\begin{aligned}&\text{(i)}\quad f \text{ takes bounded sets of } L^2(0,T;V) \text{ into bounded sets of } L^2(0,T;H);\\ &\text{(ii)}\quad f \text{ in } C^1(\mathbb{R},\mathbb{R}) \text{ such that } f' \le \varepsilon; \text{ for } \varepsilon > 0 \text{ "small."}\end{aligned} \tag{2.4}$$

Define the operator $A : X \to X^*$ by

$$\langle Au, v\rangle = \int_0^T a(t)\,dt \int_0^1 r u_r(t,r)\, v_r(t,r)\,dr + \int_0^T a(t)\,h(t)\,u(t,1)\,v(t,1)\,dt. \tag{2.5}$$

Denote by $\{w_j, j = 1,2,\dots\}$ the infinite orthonormal base in the separable Hilbert space V, and consider the following problem: Find a function $u_m(t)$ in the form

$$u_m(t) = \sum_{j=1}^{m} \xi_{mj}(t)\, w_j(r) \tag{2.6}$$

satisfying the nonlinear differential system:

$$(u_m'(t), w_j) + a(t)(u_{mr}, w_{jr}) + h(t)\,a(t)\,u_m(t,1)\,w_j(1) = (f(u_m(t)), w_j), \qquad j = 1,2,\dots,m \tag{2.7}$$

and the periodic condition

$$u_m(0) = u_m(T). \tag{2.7'}$$

It is clear that for each m there exists a solution $u_m(t)$ in the form (2.6) which satisfies (2.7) and (2.7′) almost everywhere (a.e.) on $[0, T_m]$ for some T_m, since f is locally Lipschitzian. Multiplying the j^{th} equation of the system (2.7) by ξ_j and adding these equations for $j = 1,2,\dots,m$, we have

$$\frac{1}{2}\frac{d}{dt}\|u_m(t)\|^2 + a(t)\,\|u_{mr}(t)\|^2 + a(t)\,h(t)\,u_m^2(t,1) = (f(u_m), u_m). \tag{2.8}$$

Since $V \circlearrowleft H$, it can be proved by (2.3.ii) that there exists a constant independent of t, such that the inequality

$$a(t)\,\|w_r(t)\|^2 + a(t)\,h(t)\,w^2(t,1) \ge C^1 \|w\|_V^2 \tag{2.9}$$

holds for every $w \in V$ and every $t \in \mathbb{R}$. Using the assumptions (2.4.ii), we have

$$2(f(u_m), u_m) \le (2\varepsilon + \beta)\,\|u_m\|^2 + \frac{1}{2\beta}\,|f(0)|^2, \qquad \text{for each } \beta > 0. \tag{2.10}$$

Hence, from (2.8)–(2.10), we obtain

$$\frac{d}{dt}\|u_m(t)\|^2 + (2C_1 - 2\varepsilon - \beta)\,\|u_m\|_V^2 \le \frac{1}{2\beta}\,|f(0)|^2. \tag{2.11}$$

Let us now choose ε and β such that

$$0 < \varepsilon < C_1 \text{ and } 0 < \beta < 2(C_1 - \varepsilon); \tag{2.12}$$

consequently we have since $V \circlearrowleft H$:

$$\frac{d}{dt}\|u_m(t)\|^2 + C_2\,\|u_m\|^2 \le \frac{1}{2\beta}\,|f(0)|^2, \tag{2.13}$$

with $C_2 = 2C_1 - 2\varepsilon - \beta > 0$. Integrating (2.13) we get

$$\|u_m(t)\|^2 \le R^2 + (\|u_{0m}\|^2 - R^2)\,e^{-C_2 t}, \tag{2.14}$$

where $R^2 = |f(0)|^2/2\beta C_2$ (R independent of m and t). Therefore, if we choose u_{0m} such that $\|u_{0m}\| \le R$, we get, from (2.14),

$$\|u_m(t)\| \le R, \qquad \text{i.e., } T_m = T. \tag{2.15}$$

- Let $B_m(0,R)$ be the closed ball in the space of linear combinations of the functions $w_j, j = 1,2\dots,m$, with the norm $\|\cdot\|$. Let us define $F_m : B_m(0,R) \to B_m(0,R)$ such that

$$F_m(u_{0m}) = u_m(T), \tag{2.16}$$

and show that F_m is a contraction.

Let u_{0m} and $v_{0m} \in B_m(0,R)$ and let $\Phi_m(t) = u_m(t) - v_m(t)$, where $u_m(t)$ and $v_m(t)$ are solutions of the system (2.7) on $[0,T]$ satisfying the initial conditions $u_m(0) = u_{0m}$ and $v_m(0) = v_{0m}$, respectively. $\Phi_m(t)$ satisfies the following differential equation:

$$(\Phi'_m, w_j)+a(t)(\Phi_{mr}, w_{jr})+h(t)\,a(t)\,\Phi_m(t,1)\,w_j(1) = (f(u_m)-f(v_m), w_j), j = 1,2,\dots,m \tag{2.17}$$

with the initial condition

$$\Phi_m(0) = u_{0m} - v_{0m}. \tag{2.18}$$

By using the same arguments as before, we can show that

$$\frac{d}{dt}\|\Phi_m(t)\|^2 + 2(C_1 - \varepsilon)\,\|\Phi_m\|^2 \le 0. \tag{2.19}$$

Integrating the inequality (2.19), we obtain

$$\|u_m(T) - v_m(T)\| \le e^{-(C_1-\varepsilon)T}\|u_{0m} - v_{0m}\|, \tag{2.20}$$

i.e., F_m is a contraction.

Therefore, there exists (for every m) a function $u_{0m} \in B_m(0,R)$ such that the solution of the problem (2.7) and (2.7′) is a periodic solution of the system (2.7). This solution satisfies the inequality (2.15) a.e. in $t \in [0,T]$ and consequently, by (2.11), we have

$$\int_0^T \|u_m\|_V^2\,dt \le C, \qquad C \text{ constant independent of } m. \tag{2.21}$$

- Hence, the set $\{u_m, m = 1,2,\dots\}$ is bounded in the space $L^\infty(0,T;H) \cap L^2(0,T;V)$. Further, from (2.7), we can deduce after multiplying by $\xi'_{jm}(t)$, summing up in j and integrating with respect to the time variable from 0 to T:

$$\begin{aligned}\int_0^T \|u'm(t)\|^2\,dt + \frac{1}{2}\int_0^T a(t)\,\frac{d}{dt}(\|u_{mr}\|^2)\,dt \\ + \frac{1}{2}\int_0^T a(t)\,h(t)\,\frac{d}{dt}(u_m^2(t,1))\,dt = \int_0^T (f(u_m), u'_m)\,dt.\end{aligned} \tag{2.22}$$

From (2.7′) we see that $\int_0^T (f(u_m), u'_m)\,dt = 0$. Therefore, the equality (2.22), via integration by parts, becomes

$$\int_0^T \|u'm(t)\|^2\,dt = \frac{1}{2}\int_0^T a'(t)\|u_{mr}\|^2\,dt + \frac{1}{2}\int_0^T (a(t)\,h(t))'\,u_m^2(t,1)\,dt. \tag{2.23}$$

Finally, the following inequality can be derived

$$\begin{aligned}\int_0^T \|u'_m(t)\|^2\,dt &\le \frac{1}{2}\,\|a'\|_{L^\infty(0,T)}\int_0^T \|u'_m\|_V^2\,dt + \frac{K_1}{2}\,\|(ah)'\|_{L^\infty(0,T)}\int_0^T \|u_m\|_V^2\,dt \\ &\le K_2\,\|u_m\|^2_{L^2(0,T;V)} \le K_3,\end{aligned} \tag{2.24}$$

where K_1, K_2 and K_3 are suitable constants independent of m.

- Hence, the elements of the sequence $\{u'_m, m = 1,2,\dots\}$ form a bounded set in the space $L^2(0,T;H)$.

3. EXISTENCE AND UNIQUENESS

By (2.15), (2.21) and (2.24), we can extract a subsequence of $\{u_m\}$ still denoted by $\{u_m\}$ such that

$$u_m \to u \text{ in } L^\infty(0,T;H) \text{ weakly } * \tag{3.1}$$

$$u_m \to u \text{ in } L^2(0,T;V) \text{ weakly} \tag{3.2}$$

$$u'_m \to u' \text{ in } L^2(0,T;H) \text{ weakly.} \tag{3.3}$$

From (2.7′) we get that the identity

$$u(0) = u(T) \tag{3.4}$$

holds. Denote by $\{g_i, w_i = 1, 2, \dots\}$ the orthonormal base in the real Hilbert space $L^2(0,T)$. The set $\{g_i w_j; i,j = 1,2,\dots\}$ forms an orthonormal base in $L^2(0,T;V)$. From (2.7) we have

$$((u'_m, g_i w_j)) + \langle A u_m, g_i w_j \rangle = ((f(u_m), g_i w_j)). \tag{3.5}$$

For i, j fixed, the r.h.s. of (3.5) converges to $((u', g_i w_j)) + \langle Au, g_i w_j\rangle$, $m \to \infty$. Let us now show that the l.h.s. of (3.5) converges to $((f(u), g_i w_j))$ for $m \to \infty$. Using a lemma on compactness [4] applied to (3.2) and (3.3), we can extract from the sequence $\{u_m\}$ a subsequence, still denoted by $\{u_m\}$, such that

$$u_m \to u \text{ strongly in } L^2(0,T;H). \tag{3.6}$$

By the Riesz-Fischer theorem, we can extract from $\{u_m\}$ a subsequence, denoted by $\{u_\mu\}$ such that

$$u_\mu \to u \qquad \text{a.e. in } Q =]0,T[\times]0,1[. \tag{3.7}$$

On the other hand, by (2.21) and assumption (2.4.i), we easily deduce

$$\|\sqrt{r} f(u_\mu)\|_{L^2(Q)} \le C, \qquad C \text{ constant independent of } \mu. \tag{3.8}$$

We shall now require the following lemma [4].

LEMMA 2. *Let Θ an open bounded set $\subset \mathbb{R}^N$ and g_μ, $g \in L^q(\Theta)$ $(1 < q < \infty)$ such that $\|g_\mu)\|_{L^q(\Theta)} \le C$, C constant independent of μ and $g_\mu \to g$ a.e. in Θ, then $g_\mu \to g$ weakly in $L^q(\Theta)$.*

Applying Lemma 1 with $N = q = 2$, $\Theta = Q$, $g_\mu = \sqrt{r} f(u_\mu)$, $g = \sqrt{r} f(u)$, we deduce

$$((f(u_\mu) - f(u), g_i w_j)) \to 0, \qquad \text{for } \mu \to \infty. \tag{3.9}$$

Passing to the limit in (3.5), we finally obtain

$$((u', g_i w_j)) + \langle Au, g_i w_j\rangle = ((f(u), g_i w_j)), \tag{3.10}$$

since we have (3.9). The equation (3.10) holds for every $i, j \in N$, i.e., the equation

$$((u', v)) + \langle Au, v\rangle = ((f(u), v)), \qquad \forall\, v \in L^2(0,T;V) \tag{3.11}$$

is fulfilled.

Uniqueness of the Solution

Let u_1 and u_2 be two solutions of (3.11) and (3.4). Let $w = u_1 - u_2$; then w satisfies the following problem:

$$((w', v)) + \langle Aw, v\rangle = ((f(u_1) - f(u_2), v)), \qquad \forall\, v \in L^2(0,T;V), \tag{3.12}$$

$$w(0) = w(T). \tag{3.13}$$

Taking $v = w$ in (3.12) and using (3.13) and assumption (2.4.ii), we get

$$\langle Aw, w\rangle \le \varepsilon \,\|w\|^2_{L^2(0,T;V)}. \tag{3.14}$$

Then, by virtue of (2.9), the inequality (3.14) implies

$$(C_1 - \varepsilon)\,\|w\|^2_{L^2(0,T;V)} \le 0.$$

Thus, $u_1 = u_2$, since we have $\varepsilon < C_1$. Therefore, we have the following theorem:

THEOREM 1. *The initial and boundary value problem (1.1)–(1.3) under the assumptions (2.3) and (2.4) has a unique solution $u \in L^\infty(0,T;H) \cap L^2(0,T;V)$ with derivatives $u_t \in L^2(0,T;H)$.*

4. CONTINUOUS DEPENDENCE OF THE SOLUTION

Denote by $\widetilde{X}$ the Banach space

$$\widetilde{X} = \{a \in L^\infty(0,T) \mid a' \in L^\infty(0,T)\},$$

with the norm $\|a\|_{\widetilde{X}} = \|a\|_{L^\infty(0,T)} + \|a'\|_{L^\infty(0,T)}$, and by $u = u(a,h)$ the unique solution of the Problem (3.11), (3.4). Let P_{a_0} the set defined by

$$P_{a_0} = \{a \in \widetilde{X} \mid a(t) \geq a_0 \text{ a.e. for } t \in [0,T], a(0) = a(T)\}. \tag{4.1}$$

We then have the following theorem.

THEOREM 2. *Let (2.4) hold. The solution $u(a,h)$ depends continuously of the functions a and h, i.e., if we consider $(a,h) \in P_{a_0} \times P_{h_0}$ and a sequence $\{(a_n,h_n)\} \subset P_{a_0} \times P_{h_0}$ such that $\|a_n - a\|_{L^\infty(0,T)} + \|h_n - h\|_{L^\infty(0,T)} \to 0$ for $n \to \infty$, then $\|u(a_n,h_n) - u(a,h)\|_{L^2(0,T;V)} \to 0$ for $n \to \infty$.*

PROOF. Let $(a_i,h_i) \in P_{a_0} \times P_{h_0}$; $i = 1, 2$, $u_i = u(a_i,h_i)$, $i = 1, 2$, $w = u_1 - u_2$, and consider the equation satisfied by w:

$$((w_t, v)) + \langle A_1 u_1 - A_2 u_2, v\rangle = ((f(u_1) - f(u_2), v)), \qquad \forall v \in L^2(0,T;V), \tag{4.2}$$

where the operator $A_i (i = 1,2)$ is defined by (2.5) with (a_i,h_i) instead of (a,h). Taking $v = w$ in (4.2) give

$$\langle A_1 u_1 - A_2 u_2, w\rangle = ((f(u_1) - f(u_2), w)), \tag{4.3}$$

since $w(T) = w(0)$. It is clear with the assumption (2.4) that

$$((f(u_1) - f(u_2), w)) \leq \varepsilon \|w\|^2_{L^2(0,T;V)}. \tag{4.4}$$

Considering the r.h.s. of (4.3), we have, after some rearrangements

$$\begin{aligned}\langle A_1 u_1 - A_2 u_2, w\rangle = &\int_0^T a_1(t)\,\|w_r\|^2\,dt + \int_0^T a_1 h_1\, w^2(t,1)\,dt \\ &+ \int_0^T [(a_1(t) - a_2(t))(u_{2r}, w_r) + (a_1 h_1 - a_2 h_2)\, u_2(t,1)\, w(t,1)]\,dt.\end{aligned} \tag{4.5}$$

Combining (4.3)–(4.5) and (2.9) we get

$$(C_3 - \varepsilon)\,\|w\|^2_{L^2(0,T;V)} \leq -\int_0^T \{(a_1 - a_2)(u_{2r}, w_r) + [(a_1 - a_2)\,h_1 + a_2(h_1 - h_2)]\,u_2(t,1)\,w(t,1)\}\,dt, \tag{4.6}$$

where C_3 depends only on a_0 and h_0, $C_3 - \varepsilon > 0$.

Hence, from (4.6), we obtain

$$\begin{aligned}\|u_1 - u_2\|_{L^2(0,T;V)} \leq \frac{1}{C_3 - \varepsilon}\,&[1 + K\,\|h_1\|_{L^\infty(0,T)} \\ &+ K\|a_2\|_{L^\infty(0,T)}]\,\|u_2\|_{L^2(0,T;V)}\,\left[\|a_1 - a_2\|_{L^\infty(0,T)} + \|h_1 - h_2\|_{L^\infty(0,T)}\right],\end{aligned}$$

since $|u(1)| \leq \sqrt{K}\,\|u\|_V$, K constant. ∎

5. NUMERICAL RESULTS

Now, we present some results of numerical comparison of the Bessel series representations of the solution of a nonlinear problem of the type (1.1)–(1.3) and the corresponding exact solution of this problem.

Let the problem

$$u_t - \left(u_{rr} + \frac{1}{r}\,u_r\right) = f(u) + F(t,r), \tag{5.1}$$

$$u_r(t,1) + u(t,1) = 0; \qquad u_r(t,0) = 0, \tag{5.2}$$

$$u(0,r) = u(T,r), \qquad T = 1, \tag{5.3}$$

where

$$\begin{aligned} &f(u) = \varepsilon u - u^2 \operatorname{sgn}(u),\ \varepsilon = 10^{-1}, \\ &F(t,r) = (1+\alpha r)\,e^{-\alpha r}\left[2\pi \cos 2\pi t - \varepsilon \sin 2\pi t + (1+\alpha r)\,e^{-\alpha r}(\sin 2\pi t)^2\right] \\ &\qquad + \alpha^2 e^{-\alpha r} \sin 2\pi t\,(2-\alpha r); \end{aligned}$$

$a = (1+\alpha\sqrt{5})/2$, and the domain $D = \{(t,r) \in \mathbb{R}^2 \mid 0 \le t,\ r \le 1\}$.

The exact solution of the problem (5.1)–(5.3) is

$$u(t,r) = (1+\alpha r)\,e^{-\alpha r} \sin 2\pi t.$$

To solve numerically (5.1)–(5.3), we consider the nonlinear differential system for the unknowns: $u_k(t) = u(t,r_k)$, $r_k = kh$, $h = 1/N$

$$\begin{aligned} &\frac{du_k}{dt} = \frac{1}{h^2}\left(1-\frac{1}{k}\right)u_{k-1} + \frac{1}{h^2}\left(\frac{1}{k}-2\right)u_k + \frac{u_{k+1}}{h^2} + f(u_k) + F(t,r), \\ &u_k(0) = 0, \qquad k = 1,2,\dots,N-1, \\ &u_1 = u_0, \quad u_N = \frac{u_{N-1}}{h+1}. \end{aligned} \tag{5.4}$$

To solve the nonlinear differential system (5.4) at the time t, we use the following linear recursive scheme generated by the nonlinear term $f(u_k)$:

$$\begin{aligned} &\frac{du_{k,n}}{dt} = \frac{1}{h^2}\left(1-\frac{1}{k}\right)u_{k-1,n} + \frac{1}{h^2}\left(\frac{1}{k}-2\right)u_{k,n} \\ &\qquad + \frac{u_{k+1,n}}{h^2} + f(u_{k,n-1}) + F(t,r_k), \qquad k = 1,2,\dots,N-1, \\ &u_{k,n}(0) = 0. \end{aligned} \tag{5.5}$$

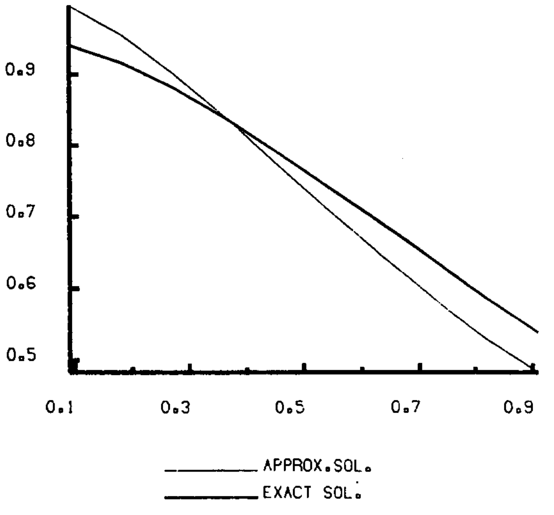

Figure 1.

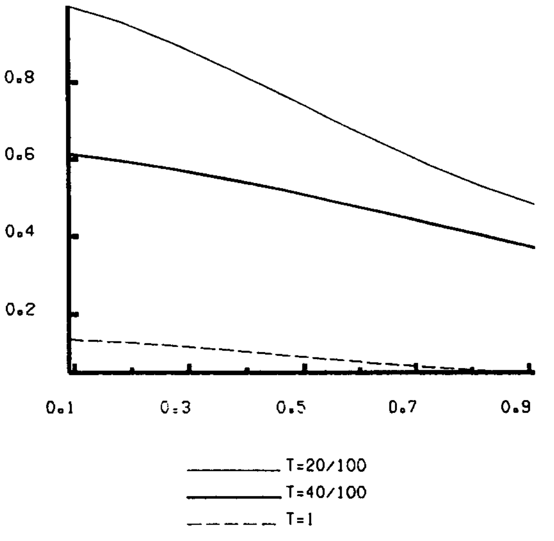

Figure 2.

The linear differential system (5.5) is solved by searching its eigenvalues and eigenfunctions. For a step $h = 1/11$, we obtained the curves in Figure 1 for the approximate solution $u_k(t)$ and the exact $\mathrm{sol}_k(t), k = 1, \ldots, 10$, and for the time $T = 10/100$. Again, with step $h = 1/11$, we obtained the curves in Figure 2 which represent the approximate solution $u_k(t)$, $k = 1, 2, \ldots, 10$ for different values of the time t : $1/5, 2/5, 1$, respectively. The numerical values at a time t are obtained by successive reinitializations from the initial time $t_0 = 0$ and for a time step $\Delta t = 1/100$.

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